

SOME NUMERICAL COMPUTATIONS ON REIDEMEISTER TORSION FOR HOMOLOGY 3-SPHERES OBTAINED BY DEHN SURGERIES ALONG THE FIGURE-EIGHT KNOT

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ABSTRACT. We show some computations on representations of the fundamental group in $SL(2; \mathbb{C})$ and Reidemeister torsion for a homology 3-sphere obtained by Dehn surgery along the figure-eight knot.

1. INTRODUCTION

In this note we show some numerical computations of representations of the fundamental group in $SL(2; \mathbb{C})$ and Reidemeister torsion for the homology 3-sphere obtained by $1/n$ -Dehn surgeries along the figure-eight knot. More precisely we enumerate all conjugacy classes of irreducible representations in $SL(2; \mathbb{C})$ and compute Reidemeister torsion for these representations.

Reidemeister torsion was originally defined by Reidemeister, Franz and de Rham in the 1930's. It is defined in more general situation, but in this paper, we consider this invariant for a homology 3-sphere M with an irreducible representation ρ of the fundamental group into $SL(2; \mathbb{C})$. It is denoted by $\tau_\rho(M) \in \mathbb{C}$.

In the 1980's Johnson [5] developed a theory of Reidemeister torsion for representations in $SU(2)$, or in $SL(2; \mathbb{C})$. That was studied from the relations to the Casson invariant. Further he proposed a torsion polynomial of a 3-manifold. In this paper, we define the torsion polynomial as follows.

Let M be a homology 3-sphere. We denote the set of conjugacy classes of representations from $\pi_1(M)$ in $SL(2; \mathbb{C})$ by $\mathcal{R}(M)$ and the subset of conjugacy classes with nontrivial value of Reidemeister torsion by $\mathcal{R}'(M)$.

Now assume that $\mathcal{R}(M) = \mathcal{R}'(M)$ and it is a finite set.

Definition 1.1. A one variable polynomial

$$\sigma_M(t) = \prod_{[\rho] \in \mathcal{R}'(M)} (t - \tau_\rho(M))$$

is called the torsion polynomial of M .

In [9, 10] we gave explicit formulas of $\sigma_M(t)$ in the case of Brieskorn homology 3-spheres obtained by surgeries along torus knots.

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Remark 1.2. The torsion polynomial was defined as

$$\pm \prod_{[\rho] \in \mathcal{R}'(M)} (t - 1/\tau_\rho(M))$$

under another normalization in [9, 10].

In this note, we show numerical computations for Dehn-surgeries along the figure-eight knot by using Mathematica.

2. SETTING

First we explain geometric setting, which is the same one in [7, 8]. Please see them for details.

Let $K \subset S^3$ be the figure-eight knot and $E(K)$ the exterior. It is well-known that $\pi_1 E(K)$ has the following presentation;

$$\pi_1 E(K) = \langle x, y \mid wx = yw \rangle$$

where $w = xy^{-1}x^{-1}y$.

One can take x as a meridian element in $\pi_1 E(K)$. As a longitude, one can do

$$l = w^{-1}\widetilde{w}$$

where $\widetilde{w} = x^{-1}yxy^{-1}$.

Let M_n be the homology 3-sphere obtained by $1/n$ -Dehn surgery along K . The fundamental group $\pi_1 M_n$ has the presentation as

$$\pi_1 M_n = \langle x, y \mid wx = yw, xl^n = 1 \rangle.$$

Let $\rho : \pi_1 M_n \rightarrow SL(2; \mathbb{C})$ be an irreducible representation. Simply we write X for $\rho(x)$, Y for $\rho(y)$ and so on. We may assume that X and Y have the following forms;

$$X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, Y = \begin{pmatrix} s & 0 \\ -t & 1/s \end{pmatrix}$$

where $s \in \mathbb{C} \setminus \{0\}$, $t \in \mathbb{C}$.

Now define the matrix R by $R = WX - YW$ where $W = XY^{-1}X^{-1}Y$. The equation $R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ induces a system of defining equations of the space of conjugacy classes of $SL(2; \mathbb{C})$ -representations of $\pi_1 E(K)$. By direct computations, we have only one equation

$$f(s, t) = 3 - \frac{1}{s^2} - s^2 + 3t - \frac{t}{s^2} - s^2 t + t^2 = 0$$

from $R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Solve $f(s, t) = 0$ in t ,

$$t = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

Hence the parameter t can be eliminated by substituting

$$t_{\pm} = \frac{1 - 3s^2 + s^4 \pm \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^2}.$$

Under $f(s, t) = 0$, there are four choices on a pair of (X, Y) as follows.

- (1) $X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, Y = \begin{pmatrix} s & 0 \\ -t_+ & 1/s \end{pmatrix}$
- (2) $X = \begin{pmatrix} s & 1 \\ 0 & 1/s \end{pmatrix}, Y = \begin{pmatrix} s & 0 \\ -t_- & 1/s \end{pmatrix},$
- (3) $X = \begin{pmatrix} 1/s & 1 \\ 0 & s \end{pmatrix}, Y = \begin{pmatrix} 1/s & 0 \\ -t_+ & s \end{pmatrix},$
- (4) $X = \begin{pmatrix} 1/s & 1 \\ 0 & s \end{pmatrix}, Y = \begin{pmatrix} 1/s & 0 \\ -t_- & s \end{pmatrix}.$

By elementary arguments in linear algebras, one can see the following.

Lemma 2.1. *Any pair of (X, Y) in the above, it gives the same conjugacy class of irreducible representations of $\pi_1 E(K)$ in $SL(2; \mathbb{C})$.*

Solve the inequality

$$1 - 2s^2 - s^4 - 2s^6 + s^8 > 0$$

under the condition $s \neq 0$ in the real numbers, one has

$$s \leq -\sqrt{\frac{1}{2}(3 + \sqrt{5})}, -\sqrt{\frac{1}{2}(3 - \sqrt{5})} \leq s < 0,$$

$$0 < s \leq \sqrt{\frac{1}{2}(3 - \sqrt{5})}, \sqrt{\frac{1}{2}(3 + \sqrt{5})} \leq s$$

and numerically

$$s \leq -1.61803, -0.618034 \leq s < 0, 0 < s \leq 0.618034, 1.61803 \leq s.$$

If s belongs to the above intervals, then the corresponding (s, t) gives an irreducible representation of $\pi_1 E(K)$ in $SL(2; \mathbb{R})$.

Here the matrix corresponding to a longitude is given by

$$L = W^{-1} \widetilde{W} = X^{-1} Y X Y^{-1} X^{-1} Y X Y^{-1}.$$

By direct computations, each entry $L_{ij}(i \leq i, j \leq 2)$ of L is given as follows:

$$L_{11} = -1 + \frac{1}{2s^4} - \frac{1}{2s^2} - \frac{s^2}{2} + \frac{s^4}{2} \pm \frac{1}{2} \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8} + \frac{\sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^4},$$

$$L_{12} = -\frac{\sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{s^3} \pm \frac{\sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{s},$$

$$L_{21} = 0,$$

$$L_{22} = -1 + \frac{1}{2s^4} - \frac{1}{2s^2} - \frac{s^2}{2} + \frac{s^4}{2} \mp \frac{1}{2} \sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8} - \frac{\sqrt{1 - 2s^2 - s^4 - 2s^6 + s^8}}{2s^4}.$$

Here the double-sign corresponds in the same order.

We consider a $1/n$ -Dehn-surgery along the figure-eight knot. Note that X is corresponding to a meridian. Then we compute the relation as

$$D = X - L^{-n} = (D_{ij}).$$

Remark 2.2. It holds that $D_{21} = 0$ identically, because X and L are upper triangular matrices in $SL(2; \mathbb{C})$.

Any solution of the systems of equations $D_{11} = D_{12} = D_{22} = 0$ gives a conjugacy class of $SL(2; \mathbb{C})$ -representation. Because X and L are upper triangular matrices, then it holds that $D_{11} = 0$ for some $s \in \mathbb{C}$ if and only if $D_{22} = 0$ for the same $s \in \mathbb{C}$. Hence we consider system of two equations $D_{11} = D_{12} = 0$.

Remark 2.3. For any solution $s_0 \in \mathbb{C} \setminus \{0\}$, then the complex conjugate of s_0 is also a solution. Because the complex conjugate $\bar{\rho}$ of ρ is always a representation for any ρ . If ρ given by s_0 is an conjugate to a representation in $SU(2)$, then ρ and $\bar{\rho}$ are conjugate each other. Because $\bar{s}_0 = 1/s_0$.

We consider Reidemeister torsion $\tau_\rho(M_n)$ for M_n with $\rho : \pi_1(M_n) \rightarrow SL(2; \mathbb{C})$. For the precise definition of Reidemeister torsion $\tau_\rho(M)$ for an $SL(2; \mathbb{C})$ -representation ρ , please see [6, 7, 11].

In the case of $1/n$ -surgeries along the figure-eight knot, we obtain the following formula of Reidemeister torsion in terms of the trace of the meridian image.

Proposition 2.4 (Kitano[9]). *If ρ is an acyclic representation, one has*

$$\tau_\rho(M) = \frac{2(u-1)}{u^2(u^2-5)} \in \mathbb{C} \setminus \{0\}$$

where $u = \text{tr}(X) = s + 1/s$.

Here ρ is called to be an acyclic representation if the chain complex of M with \mathbb{C}_ρ^2 -coefficients is an acyclic chain complex. By numerical computations we compute $\tau_\rho(M_n)$ and $\sigma_{M_n}(t)$ by using this formula.

Here we mention the Casson invariant and the $SL(2; \mathbb{C})$ -Casson invariant. Please see [1] and [4, 2, 3] for precise definitions and properties.

In 1980's Casson defined the Casson invariant $\lambda(M) \in \mathbb{Z}$ for a homology 3-sphere M as the half of algebraic count of conjugacy classes of irreducible $SU(2)$ -representations. In 2000's Curtis [4] defined the $SL(2; \mathbb{C})$ -Casson invariant $\lambda_{SL(2; \mathbb{C})}(M) \in \mathbb{Z}$ for M by counting conjugacy classes of irreducible $SL(2; \mathbb{C})$ -representations. In the case of $1/n$ -surgeries along the figure-eight knot, one has the following by applying general formula by Casson, and the one by Boden and Curtis.

Proposition 2.5.

- $\lambda(M_n) = -n$.
- $\lambda_{SL(2; \mathbb{C})}(M_n) = 4n - 1$.

Remark 2.6. The above proposition implies that the number of conjugacy classes of $SU(2)$ -representations is algebraically $2|n|$ and the one of conjugacy classes of $SL(2; \mathbb{C})$ -representations is $4|n| - 1$.

3. COMPUTATION

Here we show computations from $n = 1$ to 10 by using Mathematica. We make a list of the values of s , $u = s + 1/s$ and τ_ρ .

We remark the followings.

- We choice the value s in $|s| \leq 1$. Because the inverse $1/s$ can be done if $|s| > 1$.
- For a representation which is conjugate to the one in $SU(2)$, we choice only one value and omite its complex conjugate.

3.1. Summary. We compare computations with the values of Casson invariant $\lambda(M)$ and the $SL(2; \mathbb{C})$ -Casson invariant $\lambda_{SL(2; \mathbb{C})}(M)$. For any cases, we could find numerically $2|\lambda(M)|$ conjugacy classes in $SU(2)$. However we could find less conjugacy classes in $SL(2; \mathbb{C})$ than $|\lambda_{SL(2; \mathbb{C})}(M)|$ for $n = 7, 9, 10$.

Further for $n = 7, 9$, we could not do an $SL(2; \mathbb{R})$ -representation.

We also compute torsion polynomials $\sigma_{M_n}(t)$. We simply write $\sigma_n(t)$ to $\sigma_{M_n}(t)$. From the definition, it is a polynomial over \mathbb{Q} , because $\tau_\rho(M)$ is an algebraic number for $[\rho] \in \mathcal{R}'(M)$. All previous examples in [9, 10] are polynomials over \mathbb{Z} . Hoever they are polynomial over \mathbb{Q} , not the one over \mathbb{Z} for $n = 7, 8, 9$.

3.2. The case of $n = 1$. The first example M_1 is the Briskorn homology 3-sphere $\Sigma(2, 3, 7)$, which is not a hyperbolic manifold. Now one has

- $\lambda(M_1) = -1$,
- $\lambda_{SL(2; \mathbb{C})}(M_1) = 3$.

We find 2 conjugacy classes of $SU(2)$ -representations and totally 3 conjugacy classes of $SL(2; \mathbb{C})$ -representations. But the third one that does not come from $SU(2)$ -representations is an $SL(2; \mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.400969 + 0.916092i$	-0.801938	1.28621
◦	$0.277479 + 0.960732i$	0.554958	0.615957
	0.611406	2.24698	10.0978

In this case the torsion polynomial is given as

$$\sigma_1(t) = t^3 - 12t^2 + 20t - 8.$$

This computation coincides with the one in [9].

3.3. The case of $n = 2$. The next M_2 is a hyperbolic homology 3-sphere. One has

- $\lambda(M_2) = -2$,
- $\lambda_{SL(2; \mathbb{C})}(M_2) = 7$.

In this case we find 4 conjugacy classes of $SU(2)$ -representations and totally 7 conjugacy classes of $SL(2; \mathbb{C})$ -representations. The last representation is an $SL(2; \mathbb{R})$ -representation.

The torsion polynomial is given by

$$\sigma_2(t) = t^7 - 56t^6 + 660t^5 - 3384t^4 + 8720t^3 - 11008t^2 + 5376t - 128.$$

$SU(2)$	s	$u = s + 1/s$	τ_ρ
○	$-0.423608 + 0.905845i$	-0.847217	1.20196
○	$-0.194046 + 0.980992i$	-0.388092	3.80096
○	$0.156335 + 0.987704i$	0.31267	2.86834
○	$0.476693 + 0.87907i$	0.953386	0.0250713
	$-0.69314 \pm 0.0194149i$	$-2.13472 \mp 0.0209638i$	$2.98853 \pm 0.563052i$
	0.61642	2.23869	42.1266

3.4. **The case of $n = 3$.** Next one has

- $\lambda(M_3) = -3$,
- $\lambda_{SL(2;\mathbb{C})}(M_3) = 11$.

In this case we find 6 conjugacy classes of $SU(2)$ -representations and totally 11 conjugacy classes of $SL(2;\mathbb{C})$ -representations. The last representation is an $SL(2;\mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
○	$-0.489756 + 0.87186i$	-0.979511	1.02124
○	$-0.31143 + 0.950269i$	-0.622859	1.814
○	$-0.125581 + 0.992083i$	-0.251162	8.03486
○	$0.108419 + 0.994105i$	0.216838	6.72584
○	$0.352641 + 0.935759i$	0.705282	0.263178
○	$0.462835 + 0.886444i$	0.92567	0.0418747
	$-0.649243 \pm 0.009109i$	$-2.18919 - 0.0124968i$	$6.0064 + 1.53513i$
	$0.762562 \pm 0.0145425i$	$2.07345 - 0.010457i$	$-0.709789 + 0.0436681i$
	0.61732	2.23723	95.5058

The torsion polynomial is given by

$$\begin{aligned} \sigma_3(t) = & t^{11} - 124t^{10} + 3036t^9 - 31696t^8 + 161024t^7 - 364128t^6 \\ & + 152640t^5 + 426752t^4 - 262144t^3 - 142336t^2 + 55296t - 2048. \end{aligned}$$

3.5. **The case of $n = 4$.** Next one has

- $\lambda(M_4) = -4$,
- $\lambda_{SL(2;\mathbb{C})}(M_4) = 15$.

In this case we find 8 conjugacy classes of $SU(2)$ -representations and totally 15 conjugacy classes of $SL(2;\mathbb{C})$ -representations. The last representation is an $SL(2;\mathbb{R})$ -representation.

$$\begin{aligned} \sigma_4(t) = & t^{15} - 224t^{14} + 10320t^{13} - 211776t^{12} + 2.2964 \times 10^6 t^{11} - 1.35709 \times 10^7 t^{10} \\ & + 4.11722 \times 10^7 t^9 - 4.96721 \times 10^7 t^8 - 3.55295 \times 10^7 t^7 + 1.56351 \times 10^8 t^6 \\ & - 1.13653 \times 10^8 t^5 - 5.89578 \times 10^7 t^4 + 1.15933 \times 10^8 t^3 - 5.0004 \times 10^7 t^2 \\ & + 5.89824 \times 10^6 t - 32768 \end{aligned}$$

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.478316 + 0.878188i$	-0.956632	1.04682
◦	$-0.413993 + 0.91028i$	-0.827986	1.23604
◦	$-0.242737 + 0.970092i$	-0.485475	2.64583
◦	$-0.0926466 + 0.995699i$	-0.185293	13.9046
◦	$0.0829193 + 0.996556i$	0.165839	12.1993
◦	$0.268854 + 0.963181i$	0.537707	0.67882
◦	$0.382257 + 0.924056i$	0.764514	0.182491
◦	$0.49426 + 0.869314i$	0.98852	0.00584088
	$-0.808705 \pm 0.0102842i$	$-2.04505 - 0.00543826i$	$1.77945 + 0.0421084i$
	$-0.635184 \pm 0.00517794i$	$-2.20943 \mp 0.00765508i$	$10.2737 \pm 2.88241i$
	$0.693187 \pm 0.00964411i$	$2.13552 \mp 0.0104227i$	$-1.12125 \pm 0.112867i$
	0.617633	2.23672	170.236

3.6. **The case of $n = 5$.** Next one has

- $\lambda(M_5) = -5$,
- $\lambda_{SL(2; \mathbb{C})}(M_5) = 19$.

In this case we find 10 conjugacy classes of $SU(2)$ -representations and totally 19 conjugacy classes of $SL(2; \mathbb{C})$ -representations. The last representation is an $SL(2; \mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.496333 + 0.868132i$	-0.992666	1.00743
◦	$-0.420075 + 0.907489i$	-0.840151	1.21421
◦	$-0.343785 + 0.939049i$	-0.687569	1.57697
◦	$-0.19818 + 0.980166i$	-0.396359	3.67066
◦	$-0.073359 + 0.997306i$	-0.146718	21.4005
◦	$0.0671124 + 0.997745i$	0.13422	19.2915
◦	$0.215697 + 0.97646i$	0.431395	1.26939
◦	$0.319062 + 0.947734i$	0.638124	0.386993
◦	$0.44386 + 0.896096i$	0.88772	0.0676542
◦	$0.485869 + 0.874031i$	0.971739	0.0147589
	$-0.731102 \pm 0.00855695i$	$-2.09871 \mp 0.00744981i$	$2.35697 \pm 0.11268i$
	$-0.628893 \pm 0.00332617i$	$-2.21894 \mp 0.00508349i$	$15.7706 \pm 4.61071i$
	$0.664373 \pm 0.00643176i$	$2.16941 \mp 0.00813843i$	$-1.66792 \pm 0.199595i$
	$0.840595 \pm 0.00745097i$	$2.03014 \mp 0.00309303i$	$-0.568873 \pm 0.00810636i$
	0.617778	2.23648	266.318

The torsion polynomial is given by

$$\begin{aligned}\sigma_5(t) = & t^{19} - 348t^{18} + 24428t^{17} - 756768t^{16} + 1.22252 \times 10^7 t^{15} - 1.05049 \times 10^8 t^{14} \\ & + 4.39482 \times 10^8 t^{13} - 6.05556 \times 10^8 t^{12} - 1.45911 \times 10^9 t^{11} + 5.73225 \times 10^9 t^{10} \\ & - 3.56966 \times 10^9 t^9 - 9.32096 \times 10^9 t^8 + 1.48101 \times 10^{10} t^7 - 1.9304 \times 10^9 t^6 \\ & - 7.91541 \times 10^9 t^5 + 3.44198 \times 10^9 t^4 + 1.05592 \times 10^9 t^3 - 6.28883 \times 10^8 t^2 \\ & + 4.45645 \times 10^7 t - 524288\end{aligned}$$

3.7. **The case of $n = 6$.** One has

- $\lambda(M_6) = -6$,
- $\lambda_{SL(2;\mathbb{C})}(M_6) = 23$.

In this case we find 12 conjugacy classes of $SU(2)$ -representations and totally 23 conjugacy classes of $SL(2;\mathbb{C})$ -representations. The last representation is an $SL(2;\mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.490087 + 0.871673i$	-0.980174	1.02053
◦	$-0.460559 + 0.887629i$	-0.921118	1.0908
◦	$-0.366341 + 0.93048i$	-0.732683	1.44635
◦	$-0.290911 + 0.95675i$	-0.581822	2.00486
◦	$-0.167221 + 0.985919i$	-0.334442	4.8814
◦	$-0.0607063 + 0.998156i$	-0.121413	30.5197
◦	$0.0563605 + 0.99841i$	0.112721	28.0037
◦	$0.179677 + 0.983726i$	0.359355	2.03702
◦	$0.272216 + 0.962236i$	0.544432	0.653532
◦	$0.387688 + 0.921791i$	0.775376	0.169874
◦	$0.442457 + 0.89679i$	0.884914	0.0697032
◦	$0.497456 + 0.867489i$	0.994912	0.00256369
	$-0.863685 \pm 0.00558142i$	$-2.02147 \mp 0.00190054i$	$1.61843 \pm 0.0115852i$
	$-0.693197 \pm 0.00642162i$	$-2.13566 \mp 0.00694108i$	$3.11879 \pm 0.197207i$
	$-0.625531 \pm 0.00231384i$	$-2.22415 \mp 0.00359944i$	$22.4926 \pm 6.72158i$
	$0.64957 \pm 0.00453793i$	$2.18897 \mp 0.00621643i$	$-2.34127 + 0.304635i$
	$0.762163 \pm 0.00726544i$	$2.0741 \mp 0.00524079i$	$-0.714588 \pm 0.0221273i$
	0.617856	2.23636	383.752

The torsion polynomial is given by

$$\begin{aligned} \sigma_6(t) = & t^{23} - 504t^{22} + 52020t^{21} - 2.40364 \times 10^6 t^{20} + 5.93684 \times 10^7 t^{19} \\ & - 8.20377 \times 10^8 t^{18} + 6.23643 \times 10^9 t^{17} - 2.42844 \times 10^{10} t^{16} + 2.55758 \times 10^{10} t^{15} \\ & + 1.64156 \times 10^{11} t^{14} - 6.99939 \times 10^{11} t^{13} + 8.04666 \times 10^{11} t^{12} + 1.36418 \times 10^{12} t^{11} \\ & - 5.35777 \times 10^{12} t^{10} + 6.06942 \times 10^{12} t^9 - 6.38688 \times 10^{11} t^8 - 6.38688 \times 10^{11} t^8 \\ & - 4.81682 \times 10^{12} t^7 + 4.04757 \times 10^{12} t^6 - 2.98072 \times 10^{11} t^5 - 1.0991 \times 10^{12} t^4 \\ & + 5.29833 \times 10^{11} t^3 - 7.9675 \times 10^{10} t^2 + 3.47288 \times 10^9 t - 8.38861 \times 10^6. \end{aligned}$$

3.8. **The case of $n = 7$.** In this example, the situation is changed. First one has $\lambda(M_7) = -7$, $\lambda_{SL(2; \mathbb{C})}(M_7) = 27$.

In this case we find 14 conjugacy classes of $SU(2)$ -representations like previous examples. However we do only 22 conjugacy classes of $SL(2; \mathbb{C})$ -representations. Further we do no $SL(2; \mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
o	$-0.498132 + 0.867101i$	-0.996264	1.00376
o	$-0.456717 + 0.889612i$	-0.913434	1.10105
o	$-0.415538 + 0.909576i$	-0.831076	1.2304
o	$-0.322425 + 0.946595i$	-0.644849	1.725754883584166
o	$-0.251181 + 0.96794i$	-0.502362	2.50781
o	$-0.144537 + 0.989499i$	-0.289074	6.27537
o	$-0.0517713 + 0.998659i$	-0.103543	41.2613
o	$0.0485751 + 0.99882i$	0.0971502	38.3362
o	$0.153815 + 0.9881i$	0.30763	2.98291
o	$0.236804 + 0.971558i$	0.473608	0.982802
o	$0.340083 + 0.940396i$	0.680166	0.304734
o	$0.397537 + 0.917586i$	0.795074	0.148438
o	$0.470809 + 0.882235i$	0.941618	0.0320157
o	$0.492671 + 0.870215i$	0.985343	0.00749368
	$-0.787517 \pm 0.00610802i$	$-2.05725 \mp 0.00374012i$	$1.88117 \pm 0.0331674i$
	$-0.671806 \pm 0.00487214i$	$-2.16025 \mp 0.00592043i$	$4.04034 \pm 0.29538i$
	$0.719217 \pm 0.00597955i$	$2.10952 \mp 0.00557931i$	$-0.905083 + 0.0385034i$
	$0.881081 \pm 0.00431162i$	$2.01602 \mp 0.00124229i$	$-0.534338 \pm 0.00285536i$

Now we can see that

$$\begin{aligned} \sigma_7(t) = & t^{22} - 106.864t^{21} + 4071.84t^{20} - 70682.3t^{19} + 681805t^{18} - 4.0281 \times 10^6 t^{17} \\ & + 1.51168 \times 10^7 t^{16} - 3.5287 \times 10^7 t^{15} + 4.39339 \times 10^7 t^{14} - 1.82492 \times 10^6 t^{13} \\ & - 8.17861 \times 10^7 t^{12} + 1.07798 \times 10^8 t^{11} - 1.69554 \times 10^7 t^{10} - 8.08827 \times 10^7 t^9 \\ & + 6.5165 \times 10^7 t^8 + 5.15708 \times 10^6 t^7 - 2.53766 \times 10^7 t^6 + 7.17259 \times 10^6 t^5 \\ & + 2.28171 \times 10^6 t^4 - 1.2462 \times 10^6 t^3 + 151406t^2 - 4441.8t + 25.3003 \end{aligned}$$

and this is not a polynomial over \mathbb{Z} .

3.9. **The case of $n = 8$.** In the next case, one has

- $\lambda(M_8) = -8$,
- $\lambda_{SL(2;\mathbb{C})}(M_8) = 31$.

In this case we find 16 conjugacy classes of $SU(2)$ -representations and totally 31 conjugacy classes of $SL(2;\mathbb{C})$ -representations. The last representation is an $SL(2;\mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.494366 + 0.869254i$	-0.988732	1.01149
◦	$-0.47754 + 0.87861i$	-0.95508	1.04863
◦	$-0.419132 + 0.907925i$	-0.838263	1.21753
◦	$-0.37393 + 0.927457i$	-0.747861	1.40748
◦	$-0.28699 + 0.957934i$	-0.573979	2.04583
◦	$-0.220616 + 0.975361i$	-0.441231	3.08113
◦	$-0.127229 + 0.991873i$	-0.254459	7.85132
◦	$-0.0451269 + 0.998981i$	-0.0902538	53.6247
◦	$0.042678 + 0.999089i$	0.0853561	50.2894
◦	$0.134395 + 0.990928i$	0.268789	4.1077
◦	$0.209299 + 0.977852i$	0.418599	1.37541
◦	$0.301435 + 0.953487i$	0.60287	0.471324
◦	$0.357963 + 0.933736i$	0.715926	0.247017
◦	$0.434268 + 0.900784i$	0.868535	0.0820956
◦	$0.466323 + 0.884614i$	0.932646	0.0374964
◦	$0.49857 + 0.866849i$	0.997141	0.00143583
	$-0.894615 \pm 0.00341956i$	$-2.0124 - 0.000853032i$	$1.56556 + 0.00477245i$
	$-0.742103 \pm 0.00541306i$	$-2.08955 - 0.00441555i$	$2.23121 + 0.0587172i$
	$-0.658499 \pm 0.0037911i$	$-2.17705 - 0.00495151i$	$5.11346 + 0.407873i$
	$-0.622227 \pm 0.00130353i$	$-2.22935 - 0.0020633i$	$39.6077 + 12.0928i$
	$0.635422 \pm 0.00258348i$	$2.20915 - 0.00381495i$	$-4.06088 + 0.570821i$
	$0.693201 \pm 0.00481418i$	$2.13571 - 0.00520388i$	$-1.13212 + 0.0570251i$
	$0.808382 \pm 0.0051452i$	$2.04537 - 0.002728i$	$-0.611981 + 0.00832918i$
	0.617934	2.23623	682.674

The torsion polynomial is given by the following.

$$\begin{aligned} \sigma_8(t) = & t^{31} - 21162.9t^{30} + 2.1671 \times 10^8 t^{29} - 1.43011 \times 10^{12} t^{28} + 6.8341 \times 10^{15} t^{27} \\ & - 2.51935 \times 10^{19} t^{26} + 7.45288 \times 10^{22} t^{25} - 1.8171 \times 10^{26} t^{24} + 3.72147 \times 10^{29} t^{23} \\ & - 6.49252 \times 10^{32} t^{22} + 9.751 \times 10^{35} t^{21} - 1.27084 \times 10^{39} t^{20} + 1.44594 \times 10^{42} t^{19} \\ & - 1.4427 \times 10^{45} t^{18} + 1.26629 \times 10^{48} t^{17} - 9.79725 \times 10^{50} t^{16} + 6.68833 \times 10^{53} t^{15} \\ & - 4.02878 \times 10^{56} t^{14} + 2.13916 \times 10^{59} t^{13} - 9.99185 \times 10^{61} t^{12} + 4.09271 \times 10^{64} t^{11} \\ & - 1.46352 \times 10^{67} t^{10} + 4.54138 \times 10^{69} t^9 - 1.21315 \times 10^{72} t^8 + 2.76063 \times 10^{74} t^7 \\ & - 5.27691 \times 10^{76} t^6 + 8.31326 \times 10^{78} t^5 - 1.05097 \times 10^{81} t^4 + 1.02496 \times 10^{83} t^3 \\ & - 7.23841 \times 10^{84} t^2 + 3.29432 \times 10^{86} t - 7.25466 \times 10^{87}. \end{aligned}$$

This is also a polynomial over \mathbb{Q} , not over \mathbb{Z} , because the coefficient of t^{30} is not an integer.

3.10. **The case of $n = 9$.** In this case it is also a different situation. First one has

- $\lambda(M_9) = -9$,
- $\lambda_{SL(2; \mathbb{C})}(M_9) = 35$.

Here we find 18 conjugacy classes of $SU(2)$ -representations. But we do only 22 conjugacy classes of $SL(2; \mathbb{C})$ -representations and do no $SL(2; \mathbb{R})$ -representation.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.498871 + 0.866676i$	-0.997741	1.00227
◦	$-0.473084 + 0.881017i$	-0.946168	1.05922
◦	$-0.447444 + 0.894312i$	-0.894888	1.12697
◦	$-0.38399 + 0.923337i$	-0.76798	1.3594
◦	$-0.338051 + 0.941128i$	-0.676101	1.61426
◦	$-0.258146 + 0.966106i$	-0.516291	2.40351
◦	$-0.196502 + 0.980503i$	-0.393003	3.7226
◦	$-0.113601 + 0.993526i$	-0.227202	9.60859
◦	$-0.0399928 + 0.9992i$	-0.0799857	67.6098
◦	$0.038057 + 0.999276i$	0.0761141	63.8634
◦	$0.119296 + 0.992859i$	0.238591	5.4118
◦	$0.187397 + 0.982284i$	0.374793	1.83179
◦	$0.270042 + 0.962849i$	0.540083	0.669766
◦	$0.324317 + 0.945948i$	0.648634	0.364749
◦	$0.398186 + 0.917305i$	0.796372	0.147086
◦	$0.434658 + 0.900595i$	0.869317	0.0814873
◦	$0.482193 + 0.876065i$	0.964385	0.0188178
◦	$0.495536 + 0.868587i$	0.991072	0.00452485
	$-0.825748 \pm 0.00436242i$	$-2.03674 + 0.00203523i$	$1.71887 - 0.014448i$
	$0.905426 \pm 0.00277285i$	$2.00987 - 0.000609481i$	$-0.520584 + 0.00132641i$

The torsion polynomial is given by

$$\begin{aligned}\sigma_9(t) = & t^{22} - 164.297 t^{21} + 9092.12 t^{20} - 205717. t^{19} + 2.48075 \times 10^6 t^{18} - 1.82943 \times 10^7 t^{17} \\ & + 8.89735 \times 10^7 t^{16} - 2.97491 \times 10^8 t^{15} + 6.96286 \times 10^8 t^{14} - 1.13503 \times 10^9 t^{13} \\ & + 1.23575 \times 10^9 t^{12} - 7.72822 \times 10^8 t^{11} + 6.16538 \times 10^7 t^{10} + 3.37365 \times 10^8 t^9 \\ & - 2.79841 \times 10^8 t^8 + 7.60851 \times 10^7 t^7 + 2.06681 \times 10^7 t^6 - 2.07742 \times 10^7 t^5 \\ & + 5.86714 \times 10^6 t^4 - 724737. t^3 + 37050.3 t^2 - 579.611 t + 1.92879.\end{aligned}$$

It is clear that this is a polynomial over \mathbb{Q} , not over \mathbb{Z} .

3.11. **The case of $n = 10$.** One has

- $\lambda(M_{10}) = -10$,
- $\lambda_{SL(2;\mathbb{C})}(M_9) = 39$.

In this case we find 20 conjugacy classes of $SU(2)$ -representations. But we do only 37 conjugacy classes of $SL(2;\mathbb{C})$ -representations. The last one is an $SL(2;\mathbb{R})$ -representation.

In this case we see that the torsion polynomial is given by

$$\begin{aligned}\sigma_{10}(t) = & t^{37} - 1412.52 t^{36} + 418175 t^{35} - 5.66172 \times 10^7 t^{34} + 4.2454 \times 10^9 t^{33} \\ & - 1.90144 \times 10^{11} t^{32} + 5.3184 \times 10^{12} t^{31} - 9.80702 \times 10^{13} t^{30} + 1.24696 \times 10^{15} t^{29} \\ & - 1.12694 \times 10^{16} t^{28} + 7.34988 \times 10^{16} t^{27} - 3.4491 \times 10^{17} t^{26} + 1.12493 \times 10^{18} t^{25} \\ & - 2.2414 \times 10^{18} t^{24} + 9.72707 \times 10^{17} t^{23} + 9.31759 \times 10^{18} t^{22} - 3.10983 \times 10^{19} t^{21} \\ & + 4.21328 \times 10^{19} t^{20} + 5.21685 \times 10^{18} t^{19} - 1.18739 \times 10^{20} t^{18} + 1.91774 \times 10^{20} t^{17} \\ & - 8.61587 \times 10^{19} t^{16} - 1.47362 \times 10^{20} t^{15} + 2.57051 \times 10^{20} t^{14} - 1.15361 \times 10^{20} t^{13} \\ & - 9.11012 \times 10^{19} t^{12} + 1.35835 \times 10^{20} t^{11} - 4.44208 \times 10^{19} t^{10} - 2.62969 \times 10^{19} t^9 \\ & + 2.63319 \times 10^{19} t^8 - 5.69579 \times 10^{18} t^7 - 2.17961 \times 10^{18} t^6 + 1.4686 \times 10^{18} t^5 \\ & - 3.18791 \times 10^{17} t^4 + 3.06967 \times 10^{16} t^3 - 1.21948 \times 10^{15} t^2 + 1.60034 \times 10^{13} t \\ & - 1.36745 \times 10^{10}.\end{aligned}$$

This is also a polynomial over \mathbb{Q} , not \mathbb{Z} , because the coefficient of t^{36} is not an integer.

$SU(2)$	s	$u = s + 1/s$	τ_ρ
◦	$-0.496377 + 0.868107i$	-0.992753	1.00734
◦	$-0.48554 + 0.874215i$	-0.971079	1.03043
◦	$-0.446171 + 0.894948i$	-0.892342	1.13066
◦	$-0.416138 + 0.909302i$	-0.832276	1.22823
◦	$-0.352798 + 0.9357i$	-0.705596	1.52186
◦	$-0.307592 + 0.951518i$	-0.615185	1.84694
◦	$-0.234348 + 0.972153i$	-0.468696	2.79719
◦	$-0.177045 + 0.984203i$	-0.35409	4.43107
◦	$-0.102597 + 0.994723i$	-0.205194	11.5468
◦	$-0.035907 + 0.999355i$	-0.0718141	83.2163
◦	$0.0343385 + 0.99941i$	0.068677	79.0583
◦	$0.107229 + 0.994234i$	0.214457	6.89543
◦	$0.169577 + 0.985517i$	0.339153	2.35221
◦	$0.244255 + 0.969711i$	0.488511	0.900302
◦	$0.295848 + 0.955235i$	0.591697	0.501616
◦	$0.365525 + 0.930802i$	0.73105	0.225388
◦	$0.403589 + 0.91494i$	0.807179	0.136116
◦	$0.457052 + 0.88944i$	0.914104	0.0493696
◦	$0.478014 + 0.878352i$	0.956027	0.0235491
◦	$0.499085 + 0.866553i$	0.998171	0.000917098
	$-0.914251 \pm 0.00229081i$	$-2.00804 \mp 0.000449854i$	$1.54165 \pm 0.00241772i$
	$-0.779549 \pm 0.00431503i$	$-2.0623 \mp 0.00278537i$	$1.92757 \pm 0.0261943i$
	$-0.693202 \pm 0.00385058i$	$-2.13574 \mp 0.00416238i$	$3.12964 \pm 0.118798i$
	$-0.643412 \pm 0.00246298i$	$-2.1976 \mp 0.00348646i$	$7.70348 \pm 0.67597i$
	$-0.62071 \pm 0.000834778i$	$-2.23176 \mp 0.00133189i$	$61.6154 \pm 18.9973i$
	$-1.09378 + 0.00274066i$		
	$0.664454 \pm 0.0032119i$	$2.16941 \mp 0.00406293i$	$-1.68623 \pm 0.100765i$
	$0.731022 \pm 0.00427557i$	$2.09892 \mp 0.00372495i$	$-0.83855 \pm 0.0219205i$
	$0.84037 \pm 0.00372875i$	$2.0303 \mp 0.001551i$	$-0.569391 \pm 0.00407202i$
	0.61797	2.23617	1066.97

4. PROBLEM

In [9, 10], the torsion polynomial for a Brieskorn homology 3-sphere obtained by surgeries along a torus knot, which is not exactly same with the one given in this paper, it can be described by using Tchebychev polynomials of the first kind. It seems that it is natural, because any value of τ_ρ is given by some special values of the cosine function.

In the case of the figure-eight knots, or in more general cases of hyperbolic knots, how to treat the torsion polynomial, it is a problem.

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